

THE COHOMOLOGICAL SUPPORT LOCUS OF PLURICANONICAL SHEAVES AND THE IITAKA FIBRATION

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ABSTRACT. Let $alb_X : X \rightarrow A$ be the Albanese map of a smooth projective variety and $f : X \rightarrow Y$ the fibration from the Stein factorization of alb_X . For a positive integer m , if f and m satisfy the assumptions $AS(1, 2)$ (Sec. 1), then the translates through the origin of all components of cohomological locus $V^0(\omega_X^m, alb_X)$ generates $I^*Pic^0(S)$ where $I : X \rightarrow S$ denotes the Iitaka fibration (Theorem 1.2). This result applies to studying pluricanonical maps (Theorem 1.4, Corollary 1.7).

1. INTRODUCTION

Conventions: The varieties are assumed over \mathbb{C} and projective. If X is an irregular variety, we usually denote by $alb_X : X \rightarrow Alb(X)$ the Albanese map. A variety X with Kodaira dimension $\kappa(X) \geq 0$ has an Iitaka model X' equipped with the Iitaka fibration $I : X' \rightarrow S$, in the following we assume X always has the Iitaka fibration. A **good minimal model** of a variety is a birational model with semi-ample canonical divisor and \mathbb{Q} -factorial canonical singularities. We say a variety is good if itself is a good minimal model.

Let $a : X \rightarrow A$ be a map from a projective variety to an abelian variety, and \mathcal{F} a sheaf on X . The cohomological support locus of \mathcal{F} w.r.t. a is defined as

$$V^i(\mathcal{F}, a) := \{\alpha \in Pic^0(A) \mid h^i(\mathcal{F} \otimes a^* \alpha) \neq 0\}$$

The cohomological support locus of canonical sheaves $V^i(\omega_X, alb_X)$ plays an important role in studying irregular varieties (see [GL1], [GL2] and [CH1] and so on). In particular the cohomological support locus $V^0(\omega_X, alb_X)$ is closely related to the pluricanonical maps and Iitaka fibrations. Collecting the results of [CH2] and [JS], we have

Theorem 1.1. *Let X be a smooth irregular variety with Albanese fibers of dimension ≤ 1 and Kodaira dimension $\kappa(X) \geq 0$, and denote by $I : X \rightarrow S$ be the Iitaka fibration. Then the subgroup of $Pic^0(X)$ generated by the translates through the origin of the components of $V^0(\omega_X, alb_X)$ is $I^*Pic^0(S)$.*

With the help the result above, the authors of [Ti] and [JS] proved that if moreover X is of general type, then $\omega_X^n \otimes \alpha$ is generically very ample for every $n \geq 4$ and $\alpha \in Pic^0(X)$.

If X has higher dimensional Albanese fibers, $V^0(\omega_X, alb_X)$ may make no sense, an idea is to consider the cohomological support locus of pluri-canonical sheaves $V^0(\omega_X^m, alb_X)$ (see [CH3] and [Lai]). First we face the problem that how to choose a small m such that the locus $V^0(\omega_X^m, alb_X)$ contains “enough” information of X . For this we introduce two assumptions for a fibration $f : X \rightarrow Y$ between two normal projective varieties and a positive integer m :

- AS(1) General fibers of f have good minimal models.
- AS(2) For a general fiber F the linear system $|mK_F| \neq \emptyset$; and we can find an open set $U \subset Y$ such that, for every smooth projective curve C on Y , if $C \cap U \neq \emptyset$ (namely, C passes through a general point on Y), then $\deg(\det((f_C)_*\omega_{X_C/C}^m)) > 0$ unless $f_C : X_C \rightarrow C$ is birationally isotrivial (i.e., $\text{Var}(f_C) = 0$, see Sec. 2.4), where X_C denotes the component of the normalization of $X \times_Y C$ dominant over C and f_C denotes the natural fibration.

Our main result is

Theorem 1.2. *Let X be a smooth projective variety with $\kappa(X) \geq 0$, and denote by $I : X \rightarrow S$ the Iitaka fibration. Let $f : X \rightarrow Y$ be the fibration arising from the Stein factorization of the Albanese map $\text{alb}_X : X \rightarrow \text{Alb}(X)$. Suppose that for an integer $m > 0$, f satisfies the assumptions AS(1, 2). Then the subgroup of $\text{Pic}^0(X)$ generated by the translates through the origin of the components of $V^0(\omega_X^m, \text{alb}_X)$ is $I^*\text{Pic}^0(S)$, moreover $q(S) = q(X) - (\dim(X) - \kappa(X)) - (\dim(F) - \kappa(F))$.*

Remark 1.3. *Setting $m = 1$, AS(1, 2) are satisfied when F is a point or a curve of genus ≥ 1 ([BPV] Chap. III, Sec. 17), thus Theorem 1.1 can be obtained from Theorem 1.2 as a special case.*

Applying the (refined) result above to pluricanonical maps (see Sec. 4.5), we proved

Theorem 1.4. *Let X be a smooth projective variety, let $I : X \rightarrow S$ be the Iitaka fibration and $a = \text{alb}_S \circ I$. Let $f : X \rightarrow Y$ be the fibration arising from the Stein factorization of $a : X \rightarrow \text{Alb}(S)$ and F a general fiber. Suppose that*

- (1) AS(1, 2) are satisfied for f and $m = 1$;
- (2) for the integer $n \geq 2$, the pluricanonical map $\phi_{|nK_F|}$ is birational to the Iitaka fibration of F .

Then for every $\alpha \in \text{Pic}^0(S)$, the map $\phi_{|(n+2)K_X \otimes I^\alpha|}$ is birational to the Iitaka fibration.*

Remark 1.5. *The theorem above applies when X is of general type. If assuming F is a point or a curve of genus ≥ 3 , then we get that $\phi_{|4K_X \otimes \alpha|}$ is birational for every $\alpha \in \text{Pic}^0(X)$, i.e., the results of [Ti] and [JS] Theorem 5.3 for the case $g \geq 3$. In fact the idea of the proof is from them.*

In the setting of the theorem above, if instead assuming AS(2) is satisfied for some $m \geq 2$ or $m = 1$ in (1), then $a_\omega_X^m$ is an IT^0 sheaf by [J] Lemma 4.2 (see also Sec. 4.5) and thus $V^0(\omega_X^m, a) = \text{Pic}^0(S)$. So AS(2) always holds for $m \geq 2$ if $|mK_F| \neq \emptyset$, similarly we can prove $\phi_{|(n+2m)K_X \otimes I^*\alpha|}$ is birational to the Iitaka fibration, but this result is not optimal by the results of [CH4] Theorem 2.8.*

For the two assumptions, we remark the following known results.

- For AS(1), the existence of good minimal models is still a conjecture, but it has been proved to be true for many classes of varieties (e.g. those of general type or of maximal Albanese dimension), please refer to [Lai] for the results up to now. This assumption is needed when proving $(\text{alb}_X)_*\omega_X^m$ is a GV-sheaf.
- For AS(2), Viehweg proved that $f_*\omega_{X/Y}^m$ is weakly positive, and thus is nef ([Vie] Thm. III); and with the assumption that a general fiber F has a

good minimal model F_0 , Kawamata proved that $AS(2)$ is satisfied for some integer $m > 0$, which depends on an m_0 such that $|m_0 K_{F_0}|$ is base point free ([Ka3] Thm. 1.1, Lemma 6.1 and Sec. 7).

The theorem below tells that $AS(2)$ are satisfied for f and $m = 1$ if general fibers of f have good minimal models with trivial canonical bundles.

Theorem 1.6 (Theorem 5.1). *Let X be a smooth projective variety and $f : X \rightarrow C$ a fibration to a smooth projective curve. Suppose that general fibers of f have good minimal models with trivial canonical bundles. If $\deg(\det(f_* \omega_{X/C})) = 0$, then f is birationally isotrivial.*

Then applying Theorem 1.4, we get

Corollary 1.7. *Let X be a smooth projective variety and $I : X \rightarrow S$ the Iitaka fibration. Suppose that S has maximal Albanese dimension, and general Iitaka fiber F has a good minimal model with trivial canonical bundle. Then for every $\alpha \in \text{Pic}^0(S)$, $|4K_X \otimes I^* \alpha|$ defines a map birational to the Iitaka fibration of X .*

Remark 1.8. *In case X has maximal Albanese dimension, general fibers of its Iitaka fibration are abelian varieties, applying the corollary above we get [JMT] Theorem A(1).*

We use the approach of [CH2] Sec. 2, so the key point is the classification of the varieties with $\dim(V^0(\omega_X^m, \text{alb}_X)) = 0$ (Theorem 4.1). As an important preparation, we studied the isotrivial fibrations with $\det(f_* \omega_{X/Y}^m)$ numerically trivial. Recall that for an isotrivial fibration of a surface $f : X \rightarrow Y$ with general fibers of genus ≥ 1 , if $\det(f_* \omega_{X/Y})$ is numerically trivial, then there exists an étale cover $Z \rightarrow Y$ such that $X \times_Y Z$ is birational to $F \times Z$ ([BPV] Chap. III Prop. 18.3). We generalized this result as follows

Theorem 1.9. *Let $f : X \rightarrow Y$ be an isotrivial fibration between two smooth projective varieties and F a general fiber. Fix an integer $m > 0$. Suppose that F has a good minimal model, $h^0(\omega_F^m) > 0$ and $\det(f_* \omega_{X/Y}^m)$ is a numerically trivial line bundle. Then there exists a finite morphism $Z \rightarrow Y$ étale in codimension 1 such that $X \times_Y Z$ is birational to $F \times Z$.*

In particular, if moreover Y is a curve, then X has a birational model isogenous to a product.

Acknowledgments. *The author would like to thank Dr. Pu Cao, Prof. Yifei Chen and Hao Sun for some useful discussions, he also expresses his gratitude to Dr. Wenfei Liu and Zhi Jiang for their pointing out some fatal mistakes in the original version. The author was supported by the NSFC (No. 11226075).*

2. PRELIMINARIES

2.1. Fourier-Mukai transform. If A denotes an Abelian variety, then \hat{A} denotes its dual $\text{Pic}^0(A)$, \mathcal{P} denotes the Poincaré line bundle on $A \times \hat{A}$, and the Fourier-Mukai transform $R\Phi_{\mathcal{P}} : D^b(A) \rightarrow D^b(\hat{A})$ w.r.t. \mathcal{P} is defined as

$$R\Phi_{\mathcal{P}}(\mathcal{F}) := R(p_2)_*(Lp_1^* \mathcal{F} \otimes \mathcal{P})$$

where p_1, p_2 are the projections from $A \times \hat{A}$ to A and \hat{A} respectively. Similarly $R\Psi_{\mathcal{P}} : D^b(\hat{A}) \rightarrow D^b(A)$ is defined as

$$R\Psi_{\mathcal{P}}(\mathcal{F}) := R(p_1)_*(Lp_2^* \mathcal{F} \otimes \mathcal{P})$$

Theorem 2.1 ([Mu], Thm. 2.2). *Let A be an Abelian variety of dimension d . Then*

$$R\Psi_{\mathcal{P}} \circ R\Phi_{\mathcal{P}} = (-1)_{\hat{A}}^*[-d], \quad R\Phi_{\mathcal{P}} \circ R\Psi_{\mathcal{P}} = (-1)_{\hat{A}}^*[-d].$$

2.2. GV-Sheaves, M-regular sheaves and CGG.

Definition 2.2 ([PP], Def. 2.1, 2.2, 5.2). *Given a coherent sheaf \mathcal{F} on an Abelian variety A , its i -th cohomological support locus is defined as*

$$V^i(\mathcal{F}) := \{\alpha \in \hat{A} \mid h^i(\mathcal{F} \otimes \alpha) > 0\}$$

The number $gv(\mathcal{F}) := \min_{i>0} \{\text{codim}_{\hat{A}} V^i(\mathcal{F}) - i\}$ is called the generic vanishing index of \mathcal{F} , and we say \mathcal{F} is a GV-sheaf if $gv(\mathcal{F}) \geq 0$, and is M-regular if $gv(\mathcal{F}) > 0$, and is an IT^0 sheaf if $V^i(\mathcal{F}) = \emptyset$ for $i > 0$.

We say \mathcal{F} is continuously globally generated (CGG) if the sum of the evaluation maps

$$ev_U : \bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes \alpha) \otimes \alpha^{-1} \rightarrow \mathcal{F}$$

is surjective for any open set $U \subset \hat{A}$.

Theorem 2.3 ([Ha], Thm. 1.2, Cor. 3.2 or [PP] Sec. 2). *Let A be an Abelian variety of dimension d and \mathcal{F} a GV-sheaf on A . Then*

- (i) $\text{Pic}^0(A) \supset V^0(\mathcal{F}) \supset V^1(\mathcal{F}) \supset \dots \supset V^d(\mathcal{F})$, and $\mathcal{F} = 0$ if $V^0(\mathcal{F}) = \emptyset$;
- (ii) $R\Phi_{\mathcal{P}}(R\Delta(\mathcal{F}))[d] \cong R^d\Phi_{\mathcal{P}}(R\Delta(\mathcal{F}))$ is sheaf (where $R\Delta(\mathcal{F}) := R\mathcal{H}om(\mathcal{F}, \mathcal{O}_A)$), which we denote by $R\Delta(\mathcal{F})$;
- (iii) $\mathcal{E}xt_{\hat{A}}^i(\widehat{R\Delta(\mathcal{F})}, \mathcal{O}_{\hat{A}}) \cong (-1)_{\hat{A}}^* R^i\Phi_{\mathcal{P}}(\mathcal{F})$;
- (iv) one direct summand of \mathcal{F} is also a GV-sheaf.

Proposition 2.4 ([PP], Cor. 5.3). *An M-regular sheaf on an Abelian variety is CGG.*

2.3. Good minimal models and the cohomological support locus of ω_X^m .

On good minimal models recall that

Proposition 2.5 ([Lai] Thm. 4.5). *Let $a : X \rightarrow A$ be a morphism from a smooth projective variety to an Abelian variety. Assume that the general fibers of a have good minimal models. Then X has a good minimal model.*

Proposition 2.6. *Let $a : X \rightarrow A$ be a morphism from a smooth projective variety to an Abelian variety. Suppose that X has a good minimal model. Then for any torsion line bundle $P \in \text{Pic}(X)$ on X (i.e. $nP \equiv \mathcal{O}_X$ for some integer $n > 0$), the sheaf $a_*(\omega_X^m \otimes P)$ is a GV-sheaf on A , and every component of $V^i(a_*\omega_X^m)$, $i \geq 0$ is a translation of a sub-torus of \hat{A} via a torsion point.*

Proof. Denote by n the order of P . Consider the cyclic étale cover $\pi : X' \rightarrow X$ induced by $nP \equiv \mathcal{O}_X$. Then X' also has a good minimal model. The sheaf $\omega_X^m \otimes P$ is a direct summand of $\pi_*\omega_{X'}^m$, so $a_*(\omega_X^m \otimes P)$ is a direct summand of $a_*\pi_*\omega_{X'}^m$. If P is not trivial then considering $a \circ \pi : X' \rightarrow A$ instead, in the following we assume $P = \mathcal{O}_X$ to prove that $a_*(\omega_X^m \otimes P)$ is a GV-sheaf by Theorem 2.3 (iv).

Up to some blowing up maps, we can assume there is a morphism $\mu : X \rightarrow \bar{X}$ to one of its good minimal models. Since \bar{X} has at most canonical singularities which hence are rational, a factors through μ , so we can write that $a = \bar{a} \circ \mu$, and we have $a_*\omega_X^m = \bar{a}_*\omega_{\bar{X}}^m$ since $\mu_*\omega_X^m = \omega_{\bar{X}}^m$.

Again up to some blowing up maps, we can assume

- $K_X = \mu^* K_{\bar{X}} + \sum_i a_i E_i$ where E_i are exceptional w.r.t. μ and $a_i \geq 0$;
- there exists sufficiently divisible $N > m$ and a smooth section $D \in |\mu^* N K_{\bar{X}}|$ by Bertini's theorem, such that $D + \sum_i E_i$ is simple normal crossing.

Then for an ample divisor H on A and $t > 0$, we have

(2.1)

$$\begin{aligned}
\mu^* \omega_{\bar{X}}^m &\subseteq \mu^* \omega_{\bar{X}}^m \otimes \mathcal{O}_X \left(\sum_i [ma_i] E_i - \sum_i [(m-1)a_i] E_i \right) \\
&\subseteq \omega_X^m \otimes \mathcal{O}_X \left(- \left[\sum_i (m-1)a_i E_i \right] \right) = \omega_X^m \otimes \mathcal{O}_X \left(- \left[\frac{m-1}{N} D + \sum_i (m-1)a_i E_i \right] \right) \\
&\subseteq \omega_X^m \otimes \mathcal{I}(|(m-1)K_X|) \\
&\subseteq \omega_X^m \otimes \mathcal{I}(|(m-1)K_X + \frac{1}{t} a^* H|) \subseteq \omega_X^m
\end{aligned}$$

Pushing forward via a_* , we obtain

$$\bar{a}_* \omega_{\bar{X}}^m \subseteq a_*(\omega_X^m \otimes \mathcal{I}(|(m-1)K_X + \frac{1}{t} \tilde{a}^* H|)) \subseteq a_* \omega_X^m$$

Therefore, the two inclusions are equalities, then using [Ha] Cor. 5.2, for $i > 0, t \gg 0$ and any ample line bundle L on A , we have

$$\spadesuit H^i(a_* \omega_X^m \otimes L) = H^i(a_*(\omega_X^m \otimes \mathcal{I}(|(m-1)K_X + \frac{1}{t} \tilde{a}^* H|)) \otimes L) = 0$$

With the vanishing result above, the remaining assertions follow by the argument of [Lai] Lemma 3.4, here we give the details.

Claim 2.7. *Let L be a sufficiently ample line bundle on \hat{A} , and $\hat{L} = \Psi_{\mathcal{P}}(L)$. Then for $i > 0$, $H^i(A, a_* \omega_X^m \otimes \hat{L}^*) = 0$ where \hat{L}^* denotes the dual of \hat{L} .*

Proof of the claim. Define the isogeny $\phi_L : \hat{A} \rightarrow A$ via $\hat{a} \mapsto t_a^* L^* \otimes L$. Then $\phi_L^* \hat{L}^* \cong \oplus^{h^0(L)} L$. Let $\hat{X} = X \times_A \hat{A}$, and denote by \hat{a} and p the projections to \hat{A} and X respectively. We have

$$\begin{aligned}
H^i(A, a_* \omega_X^m \otimes \hat{L}^*) &\subseteq H^i(A, a_* \omega_X^m \otimes \hat{L}^* \otimes (\phi_L)_* \mathcal{O}_{\hat{A}}) \\
&= H^i(\hat{A}, \phi_L^*(a_* \omega_X^m \otimes \hat{L}^*)) \\
(2.2) \quad &= \oplus H^i(\hat{A}, \hat{a}_* p^* \omega_X^m \otimes L) \\
&= \oplus H^i(\hat{A}, \hat{a}_* \omega_{\hat{X}}^m \otimes L) = 0 \text{ by } \spadesuit
\end{aligned}$$

where \spadesuit applies because \hat{X} also has a good minimal model.

Then the condition [Ha] Thm 1.2(1) is satisfied, so $a_* \omega_X^m$ is a GV-sheaf. And the remaining assertion follows from [Cl-H] Thm. 8.3 and [Sim]. \square

2.4. Results on fibrations. Let $f : X \rightarrow Y$ be a fibration. We refer the readers to [Ka3] Sec. 1 for the explanation of the variation $Var(f)$, which is independent of the choices of the birational models of f . Roughly speaking $Var(f)$ is the number of moduli of fibers of f in the sense of birational geometry.

We say a fibration $f : X \rightarrow Y$ is **birationally isotrivial** if $Var(f) = 0$, which means there is a finite surjective map $\pi : \tilde{Y} \rightarrow Y$ such that $X \times_Y \tilde{Y}$ is birational to $F \times \tilde{Y}$.

Here we recall the following results.

- (1) Let X and X' be two birational projective varieties with at most canonical singularities, and let $f : X \rightarrow Y$ and $f' : X' \rightarrow Y$ be two birational fibrations. Then by considering a common resolution and comparing the push-forward of the pluricanonical sheaves, we conclude that $f_*\omega_X^m \cong f'_*\omega_{X'}^m$.
- (2) For a birationally isotrivial fibration $f : X \rightarrow Y$, if general fiber F is of Kodaira dimension $\kappa(F) \geq 0$ and has a good minimal model, then there exist a birational model $\bar{f} : \bar{X} \rightarrow Y$ with good general fibers, and an equivariant resolution $f : X' \rightarrow \bar{X} \rightarrow Y$ (see [Ka3] p.14). By [Ka3] Lemma 7.1 and Corollary 7.3, both \bar{f} and f' are isotrivial, and there exists a finite surjective map $Z \rightarrow Y$ such that the natural fibration $\bar{X} \times_Y Z \rightarrow Z$ is birational to $F \times Z \rightarrow Z$.

2.5. Results on Iitaka fibration. Let X be a smooth variety with $\kappa(X) \geq 0$, and denote by $I : X \rightarrow S$ the Iitaka fibration. Let $g : X \rightarrow Z$ be a fibration and G a general fiber. Then

- (1) every fiber of the Iitaka fibration of G is contained in some fiber of $I : X \rightarrow S$;
- (2) if g factors through I , then $\kappa(X) = \kappa(G) + \dim(Z)$;
- (3) if $\kappa(X) = 0$, then its Albanese map is a fibration onto its Albanese variety ([CH1] Theorem 1);
- (4) if X has a good minimal model, then every Iitaka fiber has non-positive Kodaira dimension.

Here we make a simple explanation for (1, 2, 4). Up to some blowing up maps we get a fibration $h : X \rightarrow W$, such that g factors through h and that the restriction map $h|_G$ coincides with its Iitaka fibration of G for general fiber G . Let F' be a general fiber of h . Assume $K_X \equiv_{\mathbb{Q}} I^*H + V$ where H is an ample \mathbb{Q} -divisor on S . By $K_{F'} \equiv K_X|_{F'}$, we have $|m(I^*H + V)|_{F'} \subseteq |mK_{F'}|$ for any $m > 0$. Since $\kappa(F') = 0$, $I^*H|_{F'} \equiv_{\mathbb{Q}} 0$. This implies F' is contained in some fiber of $I : X \rightarrow S$, hence (1) follows. If g factors through I , then (1) implies that the Iitaka fibration of G coincides with $I|_G$, thus $\kappa(G) = \dim(I(G))$, so (2) follows. For (4), take a good minimal model which we also denote by X to avoid involving too many notations. Assume that $K_X \equiv_{\mathbb{Q}} I^*H$ where H is an ample \mathbb{Q} -divisor on S . Then for every fiber F , $K_F \equiv_{\mathbb{Q}} K_X|_F \equiv_{\mathbb{Q}} I^*H|_F$ is numerically trivial, thus F has non-positive Kodaira dimension.

3. ISOTRIVIAL FIBRATION

This section is devoted to proving Theorem 1.9. We will employ Viehweg's method in [Vie] Sec. 3. For a surjective morphism $h : Y \rightarrow X$, we use three different definitions of $\omega_{Y/X}$: if X is Gorenstein and Y is Cohen-Macaulay, then $\omega_{Y/X} = \omega_Y \otimes \omega_X^{-1}$; if h is flat, locally projective and all the fibers are Cohen-Macaulay, then $\omega_{Y/X}$ is the dualizing sheaf; if h is finite, then $\omega_{Y/X}$ is the sheaf determined by $h_*\omega_{Y/X} = \mathcal{H}om(h_*\mathcal{O}_Y, \mathcal{O}_X)$.

As a preparation we introduce a lemma

Lemma 3.1. *Let X be a smooth variety and D a smooth divisor on X . For any birational morphism $f : Y \rightarrow X$, writing that $K_Y \equiv f^*K_X + \sum_i a_i E_i$ where E_i are exceptional w.r.t. f and denoting by \tilde{D} the strict transform of D w.r.t. f , then*

$$\sum_i a_i E_i + \tilde{D} \geq f^*D.$$

Proof. First we assume that both Y and \tilde{D} are smooth. By

$$K_Y + \tilde{D} \equiv f^*(K_X + D) - f^*D + \tilde{D} + \sum_i a_i E_i,$$

from adjunction formula it follows that

$$K_{\tilde{D}} = f^*K_D + (-f^*D + \tilde{D} + \sum_i a_i E_i)|_D.$$

Since D is smooth, $(-f^*D + \tilde{D} + \sum_i a_i E_i)|_D$ is effective. Write that $-f^*D + \tilde{D} + \sum_i a_i E_i = \sum_i b_i E_i$. Then we conclude that no matter whether the center of E_i intersects D or not, it must be that $b_i \geq 0$. Therefore, $-f^*D + \tilde{D} + \sum_i a_i E_i$ is effective, so we are done.

If Y or \tilde{D} is singular, then taking a resolution $\sigma : Y' \rightarrow Y$, writing that $K_{Y'} \equiv \sigma^*K_Y + \sum_j c_j F_j$ where F_j are exceptional w.r.t. σ and denoting by \tilde{D}' the strict transform of D w.r.t. $f \circ \sigma$, we find that $\tilde{D}' + \sigma^*(\sum_i a_i E_i) + \sum_j c_j F_j \geq \sigma^*(f^*D)$; pushing forward via σ_* we get $\sum_i a_i E_i + \tilde{D} \geq f^*D$. \square

First we consider the case when the base is a curve. Let $\pi_0 : C \rightarrow B$ be a Galois cover with the group G , let F be a smooth variety with a faithful action of G , and let G act on $\bar{Y} := F \times C$ diagonally. Denote by $\bar{\pi} : \bar{Y} \rightarrow \bar{X} := (F \times C)/G$ the quotient map. The singularities of \bar{X} are quotient singularities and contained in the fibers over the branch points of π_0 . Find a resolution $\mu : X \rightarrow \bar{X}$ such that the exceptional locus is contained in the fibers over the branch points of π_0 , and denote by Y the normalization of $\bar{Y} \times_{\bar{X}} X$. Then G acts naturally on Y and we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ \nu \downarrow & & \mu \downarrow \\ \bar{Y} & \xrightarrow{\bar{\pi}} & \bar{X} \end{array}$$

where the ramification locus of $\pi : Y \rightarrow X$ is contained in the exceptional locus w.r.t. ν since $\bar{\pi}$ is étale over the smooth part of \bar{X} .

Denoting by Y' the fiber product $X \times_B C$ and by g, g', f the natural fibrations from Y, Y', X to C and B respectively, we get the commutative diagram below

$$\begin{array}{ccccc} Y & \xrightarrow{\sigma} & Y' & \xrightarrow{\pi'} & X \\ g \downarrow & & g' \downarrow & & f \downarrow \\ C & \xrightarrow{id_C} & C & \xrightarrow{\pi_0} & B \end{array}$$

Theorem 3.2. *If $r = \text{rank}(f_*\omega_{X/B}^m) > 0$ and $\deg(\det(f_*\omega_{X/B}^m)) = 0$, then the cover $\pi_0 : C \rightarrow B$ is étale.*

Proof. We assume $\pi_0 : C \rightarrow B$ is ramified at the points $P_i, i = 1, 2, \dots, t$ on C by contrary, and denote by n_i the ramification order of P_i over B .

Step 1: First by ramification formula we have

$$K_Y \equiv \pi^*K_X + \sum_j b_j E_j \equiv \nu^*K_{Y'} + \sum_j a_j E_j$$

where E_j are all exceptional w.r.t. ν . Remark that if E_j is contained in the fiber over P_i , then the order of the subgroup of G fixing E_j is at most n_i , so $b_j \leq n_i - 1$.

Denote by F_i the strict transform of the fiber $F \times \{P_i\}$ w.r.t. ν . Since all the exceptional divisors E_j 's are over the P_i 's, we have

$$\diamond : g^* \sum_i (n_i - 1) P_i \geq \sum_i (n_i - 1) F_i + \sum_j b_j E_j.$$

By $K_C \equiv \pi_0^* K_B + \sum_i (n_i - 1) P_i$, we have

$$\begin{aligned} \pi^* K_{X/B} &\equiv \pi^* K_X - g^* K_C + g^* \sum_i (n_i - 1) P_i \\ &\equiv K_Y - \sum_j b_j E_j - g^* K_C + g^* \sum_i (n_i - 1) P_i \\ &\equiv \nu^* K_{\bar{Y}} + \sum_j a_j E_j - \sum_j b_j E_j - g^* K_C + g^* \sum_i (n_i - 1) P_i \\ (3.1) \quad &\equiv \nu^* K_{\bar{Y}/C} + \sum_j a_j E_j - \sum_j b_j E_j + g^* \sum_i (n_i - 1) P_i \\ &\geq \nu^* K_{\bar{Y}/C} + \sum_j a_j E_j + \sum_i (n_i - 1) F_i \dots \text{by } \diamond \\ &\equiv \nu^* K_{\bar{Y}/C} + \sum_j a_j E_j + \sum_i F_i + \sum_i (n_i - 2) F_i \\ &\geq \nu^* K_{\bar{Y}/C} + g^* \left(\sum_i P_i \right) \dots \text{by Lemma 3.1} \end{aligned}$$

Step 2: In the following we will refine the proof of [Vie] Lemma 3.3. By definition we have $\sigma_* \omega_{Y/C} = \mathcal{H}om(\sigma_* \mathcal{O}_Y, \omega_{Y'/C})$, then a natural homomorphism

$$\alpha : \sigma_* \omega_{Y/C} \rightarrow \omega_{Y'/C}$$

Since σ is affine, we get a surjection

$$\beta : \sigma^* \sigma_* \omega_{Y/C} \rightarrow \omega_{Y'/C}.$$

The homomorphism $\sigma^* \alpha : \sigma^* \sigma_* \omega_{Y/C} \rightarrow \sigma^* \omega_{Y'/C}$ factors through

$$\alpha' : \omega_{Y/C} \rightarrow \sigma^* \omega_{Y'/C}$$

By $K_Y \equiv \nu^* K_{\bar{Y}} + \sum_j a_j E_j$, we have a natural homomorphism $\nu^* \omega_{\bar{Y}/C} \rightarrow \omega_{Y/C}$, then from α' it arises

$$\bar{\alpha} : \nu^* \omega_{\bar{Y}/C} \rightarrow \sigma^* \omega_{Y'/C}$$

There exists an ideal sheaf I on Y such that $\bar{\alpha}(\nu^* \omega_{\bar{Y}/C}) = \sigma^* \omega_{Y'/C} \otimes I$. Since $\pi^* \omega_{X/B} \cong \sigma^* \omega_{Y'/C}$, by Step 1, we have a natural inclusion

$$\mathcal{O}_Y(-g^* \left(\sum_i P_i \right)) = \sigma^* g'^* \mathcal{O}_C(-\sum_i P_i) \subseteq I$$

Then we get a homomorphism

$$\gamma : \nu^* \omega_{\bar{Y}/C} \rightarrow \sigma^* (\omega_{Y'/C}(-g'^* \left(\sum_i P_i \right)))$$

Applying σ_* to $\gamma^m : \nu^* \omega_{\bar{Y}/C}^m \rightarrow \sigma^* (\omega_{Y'/C}^m(-mg'^* \left(\sum_i P_i \right)))$, we obtain

$$\sigma_* \gamma^m : \sigma_*(\nu^* \omega_{\bar{Y}/C}^m) \rightarrow \omega_{Y'/C}^m(-mg'^* \left(\sum_i P_i \right)) \subseteq \sigma_* \mathcal{O}_Y \otimes \omega_{Y'/C}^m(-mg'^* \left(\sum_i P_i \right))$$

the reason why the image is contained in $\omega_{Y'/C}^m(-mg'^*(\sum_i P_i))$ is because $\sigma_*\gamma^m$ factors through $\sigma_*\alpha'^m : \sigma_*\omega_{Y/C}^m \rightarrow \omega_{Y'/C}^m \subseteq \sigma_*\mathcal{O}_Y \otimes \omega_{Y'/C}^m$. Applying g'_* , we get a homomorphism

$$(3.2) \quad \begin{aligned} \delta : g'_*\sigma_*(\nu^*\omega_{Y/C}^m) &\cong g_*(\nu^*\omega_{Y/C}^m) \rightarrow \\ g'_*(\omega_{Y'/C}^m(-mg'^*(\sum_i P_i))) &\cong \pi_0^*f_*\omega_{X/B}^m \otimes \mathcal{O}_C(-m \sum_i P_i) \end{aligned}$$

where the second \cong is due to Y' arising from a flat base change. Noticing that δ is isomorphic except at the P_i 's and $g_*(\nu^*\omega_{Y/C}^m) \cong \oplus^r \mathcal{O}_C$ is free, we conclude that δ is injective, then a contradiction happens by $\deg(\det(\pi_0^*f_*\omega_{X/B}^m(-m \sum_i P_i))) = \deg(\det(\pi_0^*f_*\omega_{X/B}^m)) - tmr < 0$. \square

Proof of Theorem 1.9. By the results of Sec. 2.4, there exists a finite surjective morphism $\pi : Y' \rightarrow Y$ such that Y' is normal and $X \times_Y Y'$ is birational to $F \times Y'$. We can assume π is a Galois cover with group G , and there exists an action of G on F such that X is birational to $(F \times Y')/G$ where G acts on $F \times Y'$ diagonally. We can also assume G acts on F **faithfully** (Otherwise considering the subgroup $H \triangleleft G$ acting trivially on F , then $(F \times Y')/H$ is birational to $F \times (Y'/H)$).

We replace $F \times Y'$ by a smooth model X' to fit into the following commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{\pi} & Y \end{array}$$

Take a smooth curve $i : B = H_1 \cap H_2 \cap \dots \cap H_{d-1} \hookrightarrow Y$ where $d = \dim(Y)$ and the H_i 's are some general very ample divisors. We assume that the H_i 's are sufficiently general, since X', X, Y are smooth and Y' is normal, by use of Bertini's theorem $B' = \pi^{-1}B$, $X_B = X \times_Y B$ and $X'_B = X' \times_Y B$ are all smooth. We use the same notations f, f', π for the corresponding restriction maps. Note that $f_*\omega_{X_B/B}^m = i^*f_*\omega_{X/Y}^m$, hence $\deg(\det(f_*\omega_{X_B/B}^m)) = 0$. Since X'_B is birational to $F \times B'$, X_B is birational to $(F \times B')/G$. Replacing X_B by a resolution of $(F \times B')/G$ as in Theorem 3.2, it still holds that $\deg(\det(f_*\omega_{X_B/B}^m)) = 0$ by the results in Sec. 2.4, we conclude that that $\pi : B' \rightarrow B$ is étale. If $D \subset Y$ denotes the branch divisor of π , then $BD = 0$, thus $D = 0$ since B is a complete intersection of ample divisors, this means $\pi : Y' \rightarrow Y$ is étale in codimension 1. So we are done. \square

4. THE MAIN THEOREM

Notations and assumptions: We use $a : X \rightarrow A$ denoting the Albanse map and \hat{A} denoting the dual of A which is also identified with $Pic^0(X)$. Let $f : X \rightarrow Y$ be the fibration arising from the Stein factorization of a and F a general fiber, and write that $a = \pi \circ f$ where π denotes the natural map $Y \rightarrow A$. Assume that for the integer $m > 0$ and the fibration f , $AS(1, 2)$ are satisfied. Note that $V^0(a_*\omega_X^m) = V^0(\omega_X^m, a)$. Sometimes we use the notation like $V^0(a_*\omega_X^m, A)$ to mark that $V^0(\omega_X^m, a) \subset \hat{A}$.

4.1. The case when $\dim(V^0(a_*\omega_X^m)) = 0$. In this subsection $a : X \rightarrow A$ denotes a morphism to an Abelian variety which is not necessarily the Albanese map, and except this we use all the notations as above. And except for the assumptions above, suppose moreover that $\dim(V^0(a_*\omega_X^m)) = 0$.

By Proposition 2.6 $a_*\omega_X^m$ is a GV-sheaf. Then using Theorem 2.3 we have

$$R\Phi_{\mathcal{P}}(R\Delta a_*\omega_X^m) \cong R^d\Phi_{\mathcal{P}}(R\Delta a_*\omega_X^m)[-d] = \tau[-d]$$

where τ is a coherent sheaf on \hat{A} and $d = \dim(A)$; and

$$R^i\Phi_{\mathcal{P}}a_*\omega_X^m \cong \mathcal{E}xt^i(R^d\Phi_{\mathcal{P}}(R\Delta a_*\omega_X^m), \mathcal{O}_{\hat{A}}) \cong \mathcal{E}xt^i(\tau, \mathcal{O}_{\hat{A}})$$

is supported on at most finite points for every $i \geq 0$. It follows that τ is supported on finite points, thus $\mathcal{E}xt^i(\tau, \mathcal{O}_{\hat{A}}) = 0$ except when $i = d$, and then

$$R\Phi_{\mathcal{P}}a_*\omega_X^m \cong R^d\Phi_{\mathcal{P}}a_*\omega_X^m[-d]$$

and $R^d\Phi_{\mathcal{P}}a_*\omega_X^m$ is supported on finite points. Therefore, by Theorem 2.1, we have

$$a_*\omega_X^m \cong (-1)_A^* R\Psi_{\mathcal{P}} R\Phi_{\mathcal{P}}a_*\omega_X^m[d] \cong (-1)_A^* R^0\Psi_{\mathcal{P}} R^d\Phi_{\mathcal{P}}a_*\omega_X^m$$

is a numerically trivial vector bundle on A .

Theorem 4.1. *Y is an abelian variety A' étale over A , and f is isotrivial, precisely there exists an étale cover $\tilde{A} \rightarrow A'$ such that the fiber product $X \times_{A'} \tilde{A}$ is birational to $F \times \tilde{A}$.*

Proof. Similar as in the proof of Theorem 1.9 take a smooth curve $i : B = H_1 \cap H_2 \cap \dots \cap H_{d-1} \hookrightarrow A$ where the H_i 's are some general very ample divisors, and assume that $C = \pi^{-1}B$ is normal and $X_C = X \times_Y C$ is smooth. We use the same notations f, a for the corresponding restriction maps. Then $a : X_C \rightarrow B$ factors through $f : X_C \rightarrow C$, and $a_*\omega_{X_C/B}^m = i^*a_*\omega_{X/A}^m$ is numerically trivial, hence

$$\deg(\det(a_*\omega_{X_C/B}^m)) = 0$$

Since $\pi_*f_* = a_*$, we have

$$\chi(C, f_*\omega_{X_C/B}^m) = \chi(B, \pi_*f_*\omega_{X_C/B}^m) = \chi(B, a_*\omega_{X_C/B}^m)$$

and using R-R formula gives that

$$\begin{aligned} (4.1) \quad & \chi(C, f_*\omega_{X_C/B}^m) = \deg(\det(f_*\omega_{X_C/B}^m)) + r(1 - g(C)) \text{ where } r = \text{rank}(f_*\omega_{X_C/B}^m) \\ & \chi(B, a_*\omega_{X_C/B}^m) \\ & = \deg(\det(a_*\omega_{X_C/B}^m)) + \deg(\pi)r(1 - g(B)) \\ & = \deg(\pi)r(1 - g(B)) \\ & = r(1 - g(C)) + \frac{r}{2} \deg(\omega_{C/B}) \text{ by } 2g(C) - 2 = \deg(\pi)(2g(B) - 2) + \deg(\omega_{C/B}) \end{aligned}$$

thus

$$\heartsuit : \deg(\det(f_*\omega_{X_C/B}^m)) = \frac{r}{2} \deg(\omega_{C/B}).$$

On the other hand, we have

$$\deg(\det(f_*\omega_{X_C/B}^m)) = \deg(\det(f_*\omega_{X_C/C}^m \otimes \omega_{C/B}^m)) = \deg(\det(f_*\omega_{X_C/C}^m)) + rm(\deg(\omega_{C/B}))$$

Since $f_*\omega_{X_C/C}^m$ is nef, thus $\deg(\det(f_*\omega_{X_C/C}^m)) \geq rm(\deg(\omega_{C/B}))$, Then by \heartsuit it is only possible that

$$\deg(\omega_{C/B}) = 0 \text{ and } \deg(\det(f_*\omega_{X_C/C}^m)) = 0$$

thus $C \rightarrow B$ is étale. If $D \subset A$ denotes the branch divisor of π , then $BD = 0$, thus $D = 0$ since B is a complete intersection of ample divisors, this means $\pi : Y \rightarrow A$ is étale in codimension 1. It follows that $\kappa(Y) = 0$ and then Y is birational to an abelian variety A' which is étale over A (see for example [Ka1] Cor. 2), thus $Y = A'$. Moreover by the assumption $AS(2)$, X_C is birationally isotrivial over C . Take a birational model $\bar{f} : \bar{X} \rightarrow A'$ of X with good general fibers. Then for a general smooth curve C as above on A' , $\bar{X}_C = \bar{X} \times_{A'} C$ is isotrivial by results in Sec. 2.4, and so is $\bar{f} : \bar{X} \rightarrow A'$.

Take an equivariant resolution $X' \rightarrow \bar{X}$. Then the induced fibration $f' : X' \rightarrow Y$ is still isotrivial, and $f'_* \omega_{X'}^m \cong f_* \omega_X^m$ by the results in Sec. 2.4. Consider $f' : X' \rightarrow A'$ instead. Applying Theorem 1.9 we find a cover $\tilde{Y} \rightarrow A'$ unramified in codimension 1 such that $X' \times_{A'} \tilde{Y}$ is birational to $F \times \tilde{Y}$. Then $\kappa(\tilde{Y}) = 0$ and thus \tilde{Y} is an abelian variety \tilde{A} étale over A' . \square

4.2. The fibration induced by the cohomological support locus of ω_X^m . Here we denote by \hat{T} the subgroup of \hat{A} generated by the translates through the origin of the components of $V^0(\omega_X^m)$, and denote by $\iota : A \rightarrow T$ the dual map of the inclusion $\hat{T} \hookrightarrow \hat{A}$. So we can assume $V^0(a_* \omega_X^m, A) \subset \tau + \hat{T}$ where τ consists of finite torsion points on \hat{A} by Proposition 2.6. Let $a_Z \circ g : X \rightarrow Z \rightarrow T$ be the Stein factorization of $\iota \circ a$. Denote by G a general fiber of g and by K a fiber of ι . Then we get a commutative diagram

$$\begin{array}{ccccc} G & \longrightarrow & X & \xrightarrow{g} & Z \\ a \downarrow & & a \downarrow & & a_Z \downarrow \\ K & \longrightarrow & A & \xrightarrow{\iota} & T \end{array}$$

Claim 4.2. (1) G has a good minimal model.
(2) $\dim(V^0(a_* \omega_G^m, K)) = 0$.

Proof. The restriction map $f : G \rightarrow Y$ is a fibration arising from the Stein factorization of $a : G \rightarrow K$, hence the fibers have good minimal models. So (1) follows from applying Proposition 2.5.

To prove (2), consider the quotient group homomorphism $\pi : \hat{A} \rightarrow \hat{K}$ whose kernel contains some translates of \hat{T} via some torsion points, and observe that

- (a) by definition, for $\alpha \in \hat{A}$, $V^0(a_* \omega_X^m \otimes \alpha, A) + \alpha = V^0(a_* \omega_X^m, A)$, thus $V^0((a_Z \circ g)_*(\omega_X^m \otimes \alpha), T) = \hat{T} \cap V^0(a_* \omega_X^m \otimes \alpha, A) \cong (\hat{T} + \alpha) \cap V^0(a_* \omega_X^m, A)$;
- (b) $V^0(a_* \omega_X^m)$ is projected to finite torsion points $\pi(\tau)$ on \hat{K} via $\pi : \hat{A} \rightarrow \hat{K}$.

For any **torsion** point $\alpha_0 \in \hat{K}$ not contained in $\pi(\tau)$, taking a **torsion** point $\alpha \in \pi^{-1}\alpha_0$, by (a, b) above we have $V^0((a_Z \circ g)_*(\omega_X^m \otimes \alpha), T) = \emptyset$, and $(a_Z \circ g)_*(\omega_X^m \otimes \alpha)$ is a GV-sheaf by Proposition 2.6. Then we conclude that $(a_Z \circ g)_*(\omega_X^m \otimes \alpha) = 0$ by Theorem 2.3 (i), thus $V^0(a_* \omega_G^m \otimes \alpha_0, K) = 0$. Applying Proposition 2.6 to G , we conclude that $V^0(a_* \omega_G^m) \subset \pi(\tau)$, i.e., (2) is true. \square

By Step 1, we conclude

Proposition 4.3. *There exists an étale cover $\tilde{K} \rightarrow K$ such that every fiber product $G \times_K \tilde{K}$ is birational to $F \times \tilde{K}$. In particular the fiber of the Iitaka fibration of G is mapped onto K .*

4.3. The Iitaka fibration. Let $I : X \rightarrow S$ be the Iitaka fibration and F' a fiber. We get a commutative diagram

$$\begin{array}{ccccc} F' & \longrightarrow & X & \xrightarrow{I} & S \\ a \downarrow & & a \downarrow & & \text{alb}_S \downarrow \\ K' & \longrightarrow & A & \xrightarrow{\iota'} & T' \end{array}$$

Each Iitaka fiber F' has Kodaira dimension ≤ 0 , thus $V^0(a_*\omega_{F'}^m, K')$ is composed of at most finite **torsion** points, so there exists a set of **torsion** points $\tau \subset \hat{K}'$ containing **every** $V^0(a_*\omega_{F'}^m, K')$. Consider the quotient map $\pi' : \hat{A} \rightarrow \hat{K}'$ whose kernel contains some translates of $\iota'^*\hat{T}'$ via some torsion points. For any $\alpha_0 \in \hat{K}'$ not contained in τ , $V^0(a_*\omega_{F'}^m \otimes \alpha_0, K') = \emptyset$ and thus $h^0(F', \omega_{F'}^m \otimes \alpha_0) = 0$; so for every $\alpha \in \pi'^{-1}\alpha_0$, $g'_*(\omega_X^m \otimes \alpha)$ is zero, thus

$$\pi'^{-1}\alpha_0 \cap V^0(a_*\omega_X^m) = \emptyset$$

This means that $V^0(a_*\omega_X^m)$ is projected to those torsion points τ on \hat{K}' via $\pi' : \hat{A} \rightarrow \hat{K}'$, thus the kernel contains \hat{T} , i.e., $\hat{T} \trianglelefteq \iota'^*\hat{T}'$, consequently the dual map $T' \rightarrow T$ is a surjection.

4.4. Proof of Theorem 1.2. Consider the Iitaka fibration $I_G : G \rightarrow Z''$ and take a general fiber F'' . Note that F'' is contained in some F' (see Sec. 2.5). By Proposition 4.3, F'' is mapped onto K , so $K \subseteq K'$ up to a translation. Combining that $T' \rightarrow T$ is a surjection, we conclude that $K' = K$ and $T' = T$, and thus $\hat{T} = \iota'^*\hat{T}' = I^*Pic^0(S)$.

We still need to calculate $\dim(\hat{T})$. Observe that Iitaka fibration factors through $g : X \rightarrow Z$. By the results of Sec. 2.5, we have $\kappa(X) = \dim(Z) + \kappa(G)$. On the other hand, by Proposition 4.3 $\kappa(G) = \kappa(F)$. Then we are done by

$$\begin{aligned} \dim(\hat{T}) &= \dim(T) \\ &= q(X) - \dim(K) \\ &= q(X) - (\dim(G) - \dim(F)) \\ (4.2) \quad &= q(X) - (\dim(X) - \dim(Z) - \dim(F)) \\ &= q(X) - (\dim(X) - \kappa(X) + \kappa(G) - \dim(F)) \\ &= q(X) - (\dim(X) - \kappa(X)) - (\dim(F) - \kappa(F)) \end{aligned}$$

4.5. Proof of Theorem 1.4. Applying the argument above to the map $a : X \rightarrow Alb(S)$, we can prove that the translates through the origin of the components of $V^0(\omega_X, a)$ generates $Pic^0(S)$. Since X has a good minimal model, using the proof of Proposition 2.6, we have $a_*(\omega_X^k \otimes \mathcal{I}(|(k-1)K_X|)) = a_*(\omega_X^k) \neq 0$ for $k > 0$. Here we remark that

- (I) for $k \geq 2$, we have $h^i(Alb(S), a_*(\omega_X^k \otimes \mathcal{I}(|(k-1)K_X|)) \otimes Q) = 0$ for any $Q \in Pic^0(S)$ and $i > 0$ by [J] Lemma 4.2, the sheaf $a_*(\omega_X^k)$ is an IT^0 sheaf, hence is CGG by Proposition 2.4;
- (II) modifying the argument in [Ti] or [JS] Proposition 5.2, for general $x \in X$ the sheaf $a_*(I_x \otimes \omega_X^k)$ is M -regular for any $k \geq 2$ (writing that $I_x \otimes \omega_X \otimes \omega_X^{k-1}$ and using (I) if $k > 2$), hence is CGG.

Let Γ be the support of the cokernel of $a^*a_*(\omega_X) \rightarrow \omega_X$. Take two distinct points $x, y \in X$ not contained in Γ and separated by the Iitaka fibration. Suppose that x is general. We will prove that the linear system $|(n+2)K_X \otimes I^*\alpha|$ separates x and y . It suffices to find a section $s \in H^0(X, I_x \otimes \omega_X^{n+2} \otimes I^*\alpha) \neq 0$ not vanishing at y .

First consider the case $f(x) = f(y)$, i.e., x and y are contained in the same fiber F . By assumption (2) there are sections in $H^0(F, I_x \otimes \omega_F^n)$ not vanishing at y . Using (II), for any open set $U \subset \text{Pic}^0(S)$ the following composite homomorphism is surjective

$$\oplus_{\beta \in U} H^0(A, a_*(I_x \otimes \omega_X^n \otimes I^*\beta^{-1})) \otimes \beta \rightarrow a_*(I_x \otimes \omega_X^n) \rightarrow H^0(F, I_x \otimes \omega_F^n)$$

So for general $\beta \in \text{Pic}^0(S)$ we can find $t_{-\beta} \in H^0(X, I_x \otimes \omega_X^n \otimes I^*\beta^{-1})$ not vanishing at y . On the other hand by (I) for any open set $U \subset \text{Pic}^0(S)$ the following evaluation homomorphism is surjective

$$\oplus_{\beta \in U} H^0(A, a_*(\omega_X^2 \otimes I^*\alpha \otimes I^*\beta)) \otimes \beta^{-1} \rightarrow a_*(\omega_X^2 \otimes I^*\alpha)$$

And since y is not contained in Γ , the following homomorphism is surjective at y

$$a^*a_*(\omega_X^2 \otimes I^*\alpha) \rightarrow \omega_X^2 \otimes I^*\alpha$$

We conclude that for general $\beta \in \text{Pic}^0(S)$ there exists $s_\beta \in H^0(X, \omega_X^2 \otimes I^*\alpha \otimes I^*\beta)$ not vanishing at y . Then the section $t_{-\beta} \otimes s_\beta \in H^0(X, I_x \otimes \omega_X^{n+2} \otimes I^*\alpha)$ does not vanish at y .

For the case $f(x) \neq f(y)$, using (II), for any open set $U \subset \text{Pic}^0(S)$ the following evaluation map is surjective at y

$$\oplus_{\beta \in U} a^*H^0(A, a_*(I_x \otimes \omega_X^n \otimes I^*\beta^{-1})) \otimes \beta \rightarrow a^*a_*(I_x \otimes \omega_X^n) \rightarrow I_x \otimes \omega_X^n$$

Then in the same way, we can find a section $s \in H^0(X, I_x \otimes \omega_X^{n+2} \otimes I^*\alpha)$ not vanishing at y .

4.6. Problems. For $AS(2)$, we need to consider the fibrations over curves, though Kawamata gave a choice for the integer m , it is often not minimal (for example when the fibers are curves). Recall that for a fibration of a surface $f : S \rightarrow C$, if a general fiber F and C both have genus ≥ 2 then $\deg(f_*\omega_{S/C}) \geq 0$, equivalently $\chi(S) \geq \chi(F)\chi(C)$; if moreover the equality $\deg(f_*\omega_{S/C}) = 0$ is attained then f is isogenous. Here we propose the following problem, which helps to give a numerical criterion for whether a fibration is birationally isotrivial.

Problem 1. *Let $f : X \rightarrow C$ be a fibration to a curve and F a general fiber. When do the two conditions $p_g(F) > 0$ and $\deg(\det(f_*\omega_{X/C})) = 0$ imply that f is birationally isotrivial?*

5. THE FIBRATIONS WITH GENERAL FIBERS HAVING TRIVIAL CANONICAL BUNDLES

Notations and assumptions: Let X be a smooth projective variety, $f : X \rightarrow C$ a fibration to a smooth projective curve and n the dimension of general fibers. Suppose general fibers have good minimal models with trivial canonical bundles.

Theorem 5.1. *If $\deg(\det(f_*\omega_{X/C})) = 0$, then f is birationally isotrivial.*

Before the proof, we introduce two lemmas.

Lemma 5.2. *Let $C' \rightarrow C$ be a finite morphism between two smooth curves, let X' be a resolution of the fiber product $X \times_C C'$, and denote by $f' : X' \rightarrow C'$ the natural fibration. If $\deg(\det(f_*\omega_{X/C})) = 0$, then $\deg(\det(f'_*\omega_{X'/C'})) = 0$.*

Proof. This follows from [Ka3] Corollary 5.4. \square

Using [Ka2] Theorem 3 and the notations there we have

Lemma 5.3. *Let $U \subset C$ be an open set, H_0 a variation of Hodge structure with unipotent local monodromies and H the extension of H_0 on C . If for general point $t \in C$ the natural homomorphism $T_{C,t} \rightarrow \text{Hom}(F^{n,0}, F^{n-1,1})$ is injective, then $\deg(\det(F^n)) > 0$.*

Proof of Theorem 5.1. With the help of Lemma 5.2, up to a base change we can assume that over an open set $U \subset C$, the natural variation of the Hodge structure on $R^n f_* \mathbb{C}$ has unipotent local monodromies. Using Lemma 5.3, by assumption that $\deg(\det(f_*\omega_{X/C})) = 0$, for general $t \in C$ the following composite map must be zero

$$\lambda_t \circ \delta_t : T_{t,C} \rightarrow H^1(F_t, T_{F_t}) \rightarrow \text{Hom}(H^0(F_t, \Omega_{F_t}^n), H^1(F_t, \Omega_{F_t}^{n-1}))$$

where F_t denotes the fiber over t , δ_t is the Kodaira-Spencer map and $\lambda_t : H^1(F_t, T_{F_t}) \rightarrow \text{Hom}(H^0(F_t, \Omega_{F_t}^n), H^1(F_t, \Omega_{F_t}^{n-1}))$ is the period map (induced by the cup product). In the following we use the notation $\text{Hom}(H^0(F_t, \omega_{F_t}), H^1(F_t, T_{F_t} \otimes \omega_{F_t}))$ instead of $\text{Hom}(H^0(F_t, \Omega_{F_t}^n), H^1(F_t, \Omega_{F_t}^{n-1}))$.

We have a model $\bar{f} : \bar{X} \rightarrow C$ such that general fiber \bar{F}_t is a good minimal model and f factors through a resolution $\mu : X \rightarrow \bar{X}$. We will prove \bar{f} is isotrivial, so we only need to prove that for general $t \in C$ the Kodaira-Spencer map is zero

$$\bar{\delta}_t : T_{t,C} \rightarrow \text{Ext}^1(\Omega_{\bar{F}_t}^1, \mathcal{O}_{\bar{F}_t}).$$

Since $\omega_{\bar{F}_t} \equiv \mathcal{O}_{\bar{F}_t}$, the following homomorphism induced by the cup product is injective

$$\bar{\lambda}_t : \text{Ext}^1(\Omega_{\bar{F}_t}^1, \mathcal{O}_{\bar{F}_t}) \rightarrow \text{Hom}(H^0(\bar{F}_t, \omega_{\bar{F}_t}), \text{Ext}^1(\Omega_{\bar{F}_t}^1, \omega_{\bar{F}_t})).$$

(This is well known if F is a smooth variety with trivial canonical bundle, please refer to [GHJ] Sec. 16.2 whose argument applies for any dimension, and refer to [Ka3] Theorem 4.3 for more general results.)

Since \bar{F}_t has canonical singularities, so $R\mu_*\omega_{F_t} \cong \mu_*\omega_{F_t} \cong \omega_{\bar{F}_t}$ and $R\mu_*\mathcal{O}_{F_t} \cong \mathcal{O}_{\bar{F}_t}$, then by [Wa] Sec. 1.5 or [Ka3] Sec. 3 p. 12, we have two natural homomorphisms induced by $\mu^*\Omega_{\bar{F}_t}^1 \rightarrow \Omega_{F_t}^1$

$$H^1(F_t, T_{F_t}) \cong \text{Ext}^1(\Omega_{F_t}^1, \mathcal{O}_{F_t}) \rightarrow \text{Ext}^1(\Omega_{\bar{F}_t}^1, \mathcal{O}_{\bar{F}_t})$$

and

$$H^1(F_t, T_{F_t} \otimes \omega_{F_t}) \cong \text{Ext}^1(\Omega_{F_t}^1, \omega_{F_t}) \rightarrow \text{Ext}^1(\Omega_{\bar{F}_t}^1, \omega_{\bar{F}_t})$$

Then by $H^0(F_t, \omega_{F_t}) \cong H^0(\bar{F}_t, \omega_{\bar{F}_t})$, we get the following commutative diagram

$$\begin{array}{ccccc} T_{t,C} & \xrightarrow{\delta_t} & H^1(F_t, T_{F_t}) & \xrightarrow{\lambda_t} & \text{Hom}(H^0(F_t, \omega_{F_t}), H^1(F_t, T_{F_t} \otimes \omega_{F_t})) \\ id \downarrow & & \downarrow & & \downarrow \\ T_{t,C} & \xrightarrow{\bar{\delta}_t} & \text{Ext}^1(\Omega_{\bar{F}_t}^1, \mathcal{O}_{\bar{F}_t}) & \xrightarrow{\bar{\lambda}_t} & \text{Hom}(H^0(\bar{F}_t, \omega_{\bar{F}_t}), \text{Ext}^1(\Omega_{\bar{F}_t}^1, \omega_{\bar{F}_t})) \end{array}$$

As the top composite homomorphism is zero, the commutative diagram implies that the bottom one is zero too. Then the fact that $\bar{\lambda}_t$ is injective means that $\bar{\delta}_t = 0$. So we are done. \square

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